On the Coulombic scattering of a charged particle

1977 J. Phys. A: Math. Gen. 10 L79

(http://iopscience.iop.org/0305-4470/10/4/007)

J. Phys. A: Math. Gen., Vol. 10, No. 4, 1977. Printed in Great Britain.

LETTER TO THE EDITOR

On the Coulombic scattering of a charged particle

B K Ridley

Department of Physics, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, UK

Received 15 November 1976, in final form 17 January 1977

Abstract. Treating the Coulombic scattering of a charged particle by a random distribution of scatterers strictly as a two-body process, following an argument familiar in stellar dynamics, removes the divergence of the differential cross section towards small scattering angles.

In the dynamics of gaseous or solid-state plasmas one of the central problems arises out of the long-range nature of the Coulomb interaction. In the case of gaseous plasmas the conventional approach has been based on the so called BBGKY hierarchy of equations to describe the non-equilibrium statistical mechanics of plasmas, and this has led to the Balescu–Lenard equation (see for example Montgomery and Tidman 1964), in which collective effects are included under the assumption that the interactions between particles are weak and long range. If long-range, correlated motion is ignored, an approach to the problem can be made on the basis of the behaviour of a test particle which suffers successive two-body collisions (see for example Chandrasekhar 1960, Spitzer 1962). Such collisions may be roughly divided into two categories: small-impact-parameter collisions resulting in large-angle scattering (nearest neighbour); and large-impact-parameter collisions resulting in small-angle scattering (distant interactions). The latter type of collision, seen as occurring almost continuously, represents the limiting case of a stochastic process familiar in Brownian motion. Emphasizing this category leads to a Fokker–Planck equation (Gasiorowicz *et al* 1956, Rosenbluth *et al* 1957), in which screening provides a cut-off length which separates nearest-neighbour interactions (assumed to have negligible effect) from the rest. The complementary picture, in which infrequent, small-impact-parameter collisions with nearest neighbours are emphasized, leads to the Boltzmann equation, and this is the conventional model for charged-impurity scattering in semiconductors (Mott 1936, Brooks 1951, Dingle 1955), where the perturbations induced by the weak but frequent collisions with more distant neighbours are taken to be negligible, or their time average imagined to be incorporated into the electronic band structure. In both the approach in which the individual particle obeys a Fokker–Planck equation, and in the approach which leads to a Boltzmann equation, there is the common problem of limiting the range of the two-body interaction. Now it is very commonly supposed that the only acceptable way of doing this is via screening, and that the Debye length alone determines the effective range of the Coulomb potential. It is the purpose of this letter to point out that the statistics of the problem contains a criterion more basic than screening. This criterion introduces an exponential function like the screening factor into the effective collision cross section, and it therefore allows a bridge to be made between screening and non-screening situations in two-body interaction models.

The basic idea is to be found in the field of stellar dynamics (Chandrasekhar 1942, 1943) where, of course, screening does not occur. It may also be found influencing the derivation of the Fokker–Planck equation, based on the Holtsmark distribution, given by Gasiorowicz *et al* (1956). As far as the author is aware, however, the general conception has not been transparently isolated before. In order to effect such a transparency we need treat nothing more complex than the classical problem of Rutherford scattering and follow the general approach of that adopted by Chandresekhar (1943) to interpret the tail of the Holtsmark distribution. Our exponential function will be seen to arise purely out of the assumption that scatterers more distant than the nearest neighbour contribute nothing. To emphasize that our result is independent of screening, we assume the latter to be entirely absent.

Classical theory shows that the angle of deflection θ is related to the impact parameter b as follows:

$$b = R \cot(\theta/2) \tag{1}$$

where $R = Ze^2/4\pi\epsilon mv^2$. ($Ze$ = charge on centre, ϵ = permittivity, m = mass of particle, assumed to have a single elementary charge, v = its velocity.) The question is, given a density N of identical scattering centres distributed at random, what is the probability of a particle being scattered through an angle lying between θ and $\theta + \mathrm{d}\theta$? Through equation (1) we can translate that into the question: What is the probability of a scattering event having an impact parameter lying between b and $b + \mathrm{d}b$? The usual answer is to equate this with the probability of there being a scattering centre within the volume $2\pi a_0 b\ \mathrm{d}b$, where a_0 is the average distance between scattering centres. For a random distribution this probability is $2\pi N a_0 b\ \mathrm{d}b$, which is then equated to the cross section, presented by all the centres, for scattering through an angle θ into a solid angle $\mathrm{d}\Omega$. Thus:

$$Na_0\sigma(\theta)\,\mathrm{d}\Omega = 2\pi N a_0 b|\mathrm{d}b| \tag{2}$$

whence, using equation (1), we can obtain

$$\sigma(\theta) = \tfrac{1}{4}R^2 \operatorname{cosec}^4(\theta/2) \tag{3}$$

which is the standard expression for the differential cross section. (The result given by quantum mechanics in the Born approximation is identical.)

The error in this derivation is the assumption that the probability of a scattering event having an impact parameter lying between b and $b + \mathrm{d}b$ is *just* the probability of there being a scattering centre there. That is certainly a necessary condition, but it is not enough. What has been implicitly assumed is that no scattering centre with a *smaller* impact parameter intervenes. If there were, then scattering by this centre would dominate, and in the spirit of the two-body interaction approximation the more remote centre would have no effect. Thus we require the probability that there is a scattering centre with impact parameter lying between b and $b + \mathrm{d}b$ *and that there is no centre with a smaller impact parameter.* If $P(b)$ is the probability of no scattering centres with impact parameter within b, then the right-hand side of equation (2) has to be multiplied by this term, and therefore:

$$\sigma(\theta) = \tfrac{1}{4}R^2 \operatorname{cosec}^4(\theta/2)P(b). \tag{4}$$

To calculate $P(b)$ we note first of all that if p denotes the probability of there being no centre with impact parameter between b and $b + \mathrm{d}b$ then:

$$p = 1 - 2\pi N a_0 b\ \mathrm{d}b \tag{5}$$

since $2\pi Na_0 b\, db$ is the probability that such a centre exists. Then by the usual law of probabilities:

$$P(b+db)=P(b)p \tag{6}$$

and therefore it follows that:

$$P(b)=C\exp(-\pi Na_0 b^2). \tag{7}$$

Since $P(0)=1$, it follows that $C=1$. Replacing b according to equation (1) leads to the differential cross section:

$$\sigma(\theta)=\tfrac{1}{4}R^2\operatorname{cosec}^4(\theta/2)\exp[-\pi Na_0 R^2\cot^2(\theta/2)] \tag{8}$$

which no longer diverges towards small scattering angles.

The cross section is finite essentially because a third body intervenes and provides a cut-off, an effect which may be dubbed third-body interference. The consequences of third-body interference for the mobility determined by charged-impurity scattering in semiconductors will be published elsewhere.

References

Brooks H 1951 *Phys. Rev.* **83** 879

Chandrasekhar S 1942 *Principles of Stellar Dynamics* (New York: Dover) p 55ff

—— 1943 *Rev. Mod. Phys.* **15** 1

—— 1960 *Plasma Physics* (Chicago: University of Chicago Press) p 192

Dingle R B 1955 *Phil. Mag.* **46** 831

Gasiorowicz S, Neuman M and Riddell R J 1956 *Phys. Rev.* **101** 922

Montgomery D C and Tidman D A 1964 *Plasma Kinetic Theory* (New York: McGraw-Hill)

Mott N F 1936 *Proc. Camb. Phil. Soc.* **32** 281

Rosenbluth M, MacDonald W M and Judd D L 1957 *Phys. Rev.* **107** 1

Spitzer L 1962 *Physics of Fully Ionised Gases* (New York: Interscience) p 127